APPLICABILITY CRITERIA FOR KINEMATIC AND DIFFUSION ROUTING MODELS[1]

INTRODUCTION

Many simplified flood routing models may be categorized as either kinematic-type or diffusion-type models. The Muskingum Model (McCarthy, 1938), Reservoir Routing Model (Goodrich, 1931), SSARR Model (Rockwood, 1958), Kinematic Model (Lighthill and Whitham, 1955), and the SWMM Model (Huber et al., 1975) are kinematic-type models. The Muskingum-Cunge Model (Cunge, 1969) is a diffusion-type model. These models are limited to applications where the inertial effects are insignificant and, in the case of the kinematic-type models, the water surface slope is constant with time and is closely approximated by the channel bottom slope.

This paper develops criteria to quantify the acceptable range of application for kinematic and diffusion models. The criteria are developed by estimating the magnitude of the terms in the conservation of momentum equation which are neglected by the kinematic and diffusion models. The omitted terms are normalized with the channel bottom slope; this ratio is expressed with hydraulic variables (channel bottom slope, peak discharge, Manning n, time of rise of inflow hydrograph, cross-section parameters) whose values are readily available prior to routing. The criteria are applicable for a wide range of practical channel shapes and typical inflow hydrograph shapes.

Ponce et al. (1978) also presented criteria for selecting appropriate applications for kinematic and diffusion models. Their results, which were obtained for a sinusoidal shaped wave in a wide channel by using a linear analysis technique, are compared with those developed herein. Realistic hydrograph shape, cross-section shape, nonlinearity and non-prismatic channel characteristics are considered in the approach presented in this paper.

Kinematic and diffusion models are also limited to applications where insignificant backwater effects exist and where wave propagation is in the downstream flow direction only. No attempt is made herein to quantify these restrictions.

All notation is defined in Appendix A--Notation.

THEORETICAL DEVELOPMENT

The conservation of momentum of one-dimensional unsteady flow is described by the following:

$$\partial V/\partial t + V\,\partial V/\partial x + g(\partial y/\partial x - S_o + S) = 0 \qquad (1)$$

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Solving Eq. (1) for the friction slope (S) and then dividing through by the channel bottom slope (S_o), one obtains:

$$\frac{S}{S_o} = \overset{(1)}{1} - \overset{(2)}{\frac{\partial y/\partial x}{S_o}} - \frac{V\,\partial V/\partial x}{g\,S_o} - \overset{(3)}{\frac{\partial V/\partial t}{g\,S_o}} \tag{2}$$

The conservation of mass of one-dimensional unsteady flow is described by the following:

$$A\,\partial V/\partial x + V\,\partial A/\partial x + B\,\partial y/\partial t = 0 \tag{3}$$

Also, the term ($\partial A/\partial x$) may be approximated as:

$$\partial A/\partial x = B\,\partial y/\partial x + y\,\partial B/\partial x \tag{4}$$

and the hydraulic depth (D) is by definition,

$$D = A/B \tag{5}$$

Now, solving Eq. (3) for $\partial V/\partial x$, the following is obtained:

$$\partial V/\partial x = -V/A\,\partial A/\partial x - B/A\,\partial y/\partial t \tag{6}$$

Upon substituting Eqs. (4) and (5) into Eq. (6) and then multiplying through by V, the following expression for the term ($V\,\partial v/\partial x$) is obtained:

$$V\,\partial V/\partial x = -V^2/D\,\partial y/\partial x - V^2/B\,\partial B/\partial x - V/D\,\partial y/\partial t \tag{7}$$

Using the kinematic approximation for $\partial y/\partial x$ (Henderson, 1966), i.e.,

$$\partial y/\partial x = -1/c\,\partial y/\partial t \tag{8}$$

where c is the kinematic wave speed which can be evaluated using the following:

$$c = KV \tag{9}$$

in which K is a cross-sectional shape factor, $7/6 \leq K \leq 5/3$. Hence, by substituting Eq. (9) into Eq. (8), the expression for $\partial y/\partial x$ becomes:

$$\partial y/\partial x = -\partial y/\partial t/(KV) \tag{10}$$

Returning now to Eq. (2), the following is obtained by substituting Eqs. (7) and (10) into Eq. (2):

$$\frac{S}{S_o} = \overset{(1)}{1} + \overset{(2)}{\frac{\partial y/\partial t}{K\,V\,S_o}} + \frac{1}{g\,S_o}\left[\frac{V^2}{B}\,\partial B/\partial x + (1-1/K)\,\frac{V}{D}\,\partial y/\partial t - \overset{(3)}{\partial V/\partial t}\right] \tag{11}$$

Let the time derivatives be represented by the following expressions:

$$\partial y/\partial t = M\ y_p/T \tag{12}$$

$$\partial V/\partial t = M\ V_p/T \tag{13}$$

in which y_p is the peak depth, V_p is the peak velocity, and M is a multiplier which adjusts the straight line approximation for the rising limb of the hydrograph to that maximum slope associated with the rising limb of a hydrograph having a gamma distribution. An expression for M is developed by differentiating the gamma function, evaluating this expression at $t = 2T/3$ and then forming a ratio of the evaluated expression to the straight line approximation. (See Appendix B for the derivation of M). The resulting expression for M is:

$$M = \alpha/2\ (2/3)^{\alpha}\ e^{\alpha/3} \tag{14}$$

where: $$\alpha = 1/(T_g/T_r - 1) \tag{15}$$

in which T_g is the time (hr.) from beginning of rise to the center of gravity of the hydrograph, and T_r is the time (hr.) from the beginning of rise to the peak of the hydrograph. The minimum value for M is one when the rising limb is a straight line; and its maximum value of about 2.5 occurs when T_g/T_r is 1.09 and $t = 2T_r/3$. Thus, $1 < M < 2.5$ represents the range for the multiplier parameter (M).

It is more convenient if the time of rise (T_r) is expressed in hrs.; therefore, Eqs. (12-13) become:

$$\partial y/\partial t = M\ y_p/(3600\ T_r) \tag{16}$$

$$\partial V/\partial t = M\ V_p/(3600\ T_r) \tag{17}$$

in which T_r is the time of rise (hr.) of the hydrograph.

The velocity (V) in the preceding equations is assumed to be the velocity occurring when $t = 2T_r/3$, i.e.,

$$V = 2\ V_p/3 \tag{18}$$

since the velocity at the beginning of rise is assumed negligible compared to V_p. Upon substituting Eqs. (16-18) into Eq. (11) and simplifying, the following is obtained:

$$\frac{S}{S_o} = \underset{(1)}{1} + \underset{(2)}{\frac{0.000417\ M\ y_p}{T_r\ S_o\ K\ V_p}} + \underset{(3)}{\frac{0.00000863\ M\ V_p}{T_r\ S_o}} \left[\beta + (1-1/K)\ \frac{2\ y_p}{3D} - 1\right] \tag{19}$$

where: $$\beta = \frac{1600\ T_r\ V_p}{M\ B}\ \Delta\bar{B}/\Delta x \tag{20}$$

REPRESENTATION OF HYDRAULIC PARAMETERS

In order to evaluate the terms in Eq. (19) before routing an inflow hydrograph, it is necessary to express the parameters (y_p, V_p, K, D, B) in terms of parameters which are known a priori. To accomplish this, as well as account for the effect of cross-sectional shape, the channel geometry is approximated as:

$$B = k\, y^m \tag{21}$$

$$A = \frac{k\, y^{m+1}}{m+1} \tag{22}$$

$$D = A/B = y/(m+1) \tag{23}$$

in which k and m are fitted parameters for the observed variation of B with y. Scaling is accomplished via k, and m accounts for the shape. Rectangular (wide channel), parabolic, and triangular shaped channels have m values of 0, 0.5, and 1.0, respectively. A value of $m > 1$ represents an expanding V-shape section in which the width (B) increases at a nonlinear rate with depth (y). This shape is appropriate for many natural cross sections composed of a relatively narrow in-bank channel and a rather wide over-bank (floodplain) section.

Using the Manning equation, the unit-width peak discharge (q_p) is given by:

$$q_p = 1.49/n\; S_o^{1/2}\; D_p^{5/3} \tag{24}$$

Then, substituting Eq. (23) into Eq. (24) and solving for y_p, the following is obtained:

$$y_p = (q_p/a)^{0.6} \tag{25}$$

where: $a = 1.49\; S_o^{1/2}/[n\; (m+1)^{5/3}]$ (26)

Also, the unit-width discharge (q_p) can be expressed as Q_p/B or

$$q_p = a \left(\frac{Q_p}{k\, a}\right)^{5/(3m+5)} \tag{27}$$

in which Q_p is the peak discharge of the inflow hydrograph.

Using the Manning equation, the peak velocity (V_p) can be expressed as:

$$V_p = 1.49/n\; S_o^{1/2}\; D_p^{2/3} \tag{28}$$

Substituting peak values for V and D (using Eq. (23) for the latter) results in the following:

$$V_p = 1.49/n\; S_o^{1/2}\; [y_p/(m+1)]^{2/3} \tag{29}$$

Now, substituting Eq. (25) into Eq. (29) yields an expression for V_p:

$$V_p = (m+1)\, a^{0.6}\, q_p^{0.4} \tag{30}$$

The ratio (V_p/y_p) can be obtained from Eq. (25) and (30). Thus,

$$V_p/y_p = (m+1)\, a^{1.2}/q_p^{0.2} \tag{31}$$

The cross-sectional factor (K) in Eq. (9) is given by the following:

$$K = 5/3 - 2/3\, dB/dy\, A/B^2 \tag{32}$$

Upon substituting Eqs. (21-22) into Eq. (32) and simplifying, the following is obtained:

$$K = (3m+5)/[3(m+1)] \tag{33}$$

The ratio $[2\, y_p/(3D)]$ in Eq. (19) may be obtained as follows:

$$2\, y_p/(3D) = \frac{2\, y_p}{3(2\, y_p/3)/(m+1)} = m+1 \tag{34}$$

Then, the term in brackets of Eq. (19) can be evaluated using Eqs. (33), and (34). Thus,

$$\beta + (1-1/K)\, 2\, y_p/(3D) - 1 = \beta - (m+3)/(3m+5) \tag{35}$$

The ratio (V_p/B) is obtained using Eqs. (21) and (30). Thus,

$$V_p/B = 1.5^m (m+1) a^{0.6(m+1)}/(k\, q_p^{0.6m-0.4}) \tag{36}$$

Then, using Eq. (36), the nonprismatic term (β), Eq. (20), can be expressed as follows:

$$\beta = \frac{1600\, T_r\, 1.5^m (m+1)\, a^{0.6(m+1)}}{M\, k\, q_p^{0.6m - 0.4}}\, \Delta\bar{B}/\Delta x \tag{37}$$

in which $\Delta\bar{B}/\Delta x$ is the average variation of the width (B) along the routing reach (Δx); this is the nonprismatic characteristic of the routing reach.

DIFFUSION ROUTING MODELS

Diffusion-type routing models are based on the following approximation for S:

$$S = S_o - \partial y/\partial x \tag{38}$$

Therefore, only the inertial effect represented by the third (3) term in Eqs. (2), (11), and (19) is omitted. The error due to the omission of this term is denoted as E_r, expressed as a decimal fraction. The following inequality expresses the relationship that must exist if the omitted term should not cause a relative error in the conservation of momentum equation greater than E_r:

$$E_r > \frac{0.00000863\ M\ V_p}{T_r\ S_o} [\beta + (1-1/K) 2\ y_p/(3D) - 1] \tag{39}$$

Using Eqs. (30) and (35), Eq. (39) can be expressed as follows:

$$E_r > \frac{0.000011\ M\ q_p^{0.4}}{T_r\ S_o^{0.7}\ n^{0.6}} |\beta - (m+3)/(3m+5)| \tag{40}$$

Replacing E_r with E/100, in which E is in percent, Eq. (40) can be rearranged as follows:

$$\frac{T_r\ S_o^{0.7}\ n^{0.6}}{M\ \Phi'\ q_p^{0.4}} > 0.0011/E \tag{41}$$

where: $$\Phi' = |\beta - (m+3)/(3m+5)| \tag{42}$$

In Eq. (42), E is a quantitative index related to the maximum error (percent) that is tolerated when the inequality is satisifed. When the channel is prismatic ($\Delta\bar{B}/\Delta x = 0$) with rectangular cross-section (m=0) and assuming M = 2.5, Eq. (41) becomes:

$$\frac{T_r\ S_o^{0.7}\ n^{0.6}}{q_p^{0.4}} > 0.0017/E \tag{43}$$

KINEMATIC ROUTING MODELS

Kinematic-type routing models are based on the assumption of a single-valued depth-discharge relation, i.e.,

$$\partial A/\partial Q = dA/dQ = 1/c \tag{44}$$

Eq. (44) implies that the friction slope (S) is constant and equal to the bottom slope, i.e.,

$$S = S_o \tag{45}$$

or $$S/S_o = 1 \tag{46}$$

Therefore, kinematic models omit the second (2) and third (3) terms of Eqs. (2), (11), and (19). The error due to the omission of these terms is denoted as E_r expressed as a decimal fraction. The following inequality expresses the relationship that must exist if the omitted terms should not cause a relative error of conservation of momentum greater than E_r:

$$E_r > \frac{0.000417\ M\ y_p}{T_r\ S_o\ K\ V_p} + \frac{0.00000863\ M\ V_p}{T_r\ S_o} [\beta + (1-1/K)\ 2\ y_p/(3D) - 1] \tag{47}$$

Substituting Eqs. (30), (31), (33), (35), and (42) into Eq. (47), the following is obtained:

$$E_r > \frac{0.000777\ M\ q_p^{0.2}\ n^{1.2}\ (m+1)^2}{T_r\ S_o^{1.6}\ (3m+5)} + \frac{0.000011\ M\ q_p^{0.4}\ \Phi'}{T_r\ S_o^{0.7}\ n^{0.6}} \tag{48}$$

Replacing E_r with E/100, in which E is in percent, Eq. (48) can be rearranged in the following form:

$$E > \frac{0.0777\ M\ q_p^{0.2}\ n^{1.2}\ \Phi\ (1+I)}{T_r\ S_o^{1.6}} \tag{49}$$

where: $\Phi = (m+1)^2/(3m+5)$ (50)

$$I = 0.014\ S_o^{0.9}\ q_p^{0.2}\ \Phi'/(\Phi\ n^{1.8}) \tag{51}$$

The parameter (I) acounts for the term (3) in Eq. (47). It represents less than about 17% of term (2) for flows with Froude numbers (F) less than 0.5 and only represents 4% of term (2) when the Froude number is 0.25. The parameter (I) can also be expressed in terms of the Froude number (F), i.e., $I = 0.22\ F^2\ \Phi'/\Phi$.

Rearranging Eq. (49), the following criterion is obtained for kinematic models:

$$\frac{T_r\ S_o^{1.6}}{M\ (1+I)\ \Phi\ q_p^{0.2}\ n^{1.2}} > 0.078/E \tag{52}$$

If the inequality in Eq. (52) is satisfied, the maximum error that will be incurred using a kinematic model is E (percent). If the channel is prismatic, rectangular, M is assumed to be 2.5, and the Froude number is about 0.5 such that I may be approximated as 0.15, Eq. (52) reduces to the following:

$$\frac{T_r\ S_o^{1.6}}{q_p^{0.2}\ n^{1.2}} > 0.045/E \tag{53}$$

COMPARISON WITH CRITERIA OF PONCE AND SIMONS

Ponce and Simons (1977) developed an analytical solution to a linearized version of the unsteady flow equations, Eqs. (1) and (3), for a sinusoidal shaped wave propagating in a wide channel. Ponce et al. (1978) used this information to define the limits of applicability of the kinematic and diffusion type models.

<u>Kinematic Criterion</u>. In the case of kinematic models, they presented the following criterion for routing errors at the 5% level:

$$\frac{T_D\ S_o\ V_o}{D_o} > 171 \tag{54}$$

in which T_D is the duration (sec.) of the inflow hydrograph. Using Eqs. (23), and (30) and the approximation that $T_D = (2)(3600)T_r$, Eq. (54) may be rewritten in the following equivalent form:

$$\frac{T_r\, S_o^{1.6}}{q_p^{0.2}\, n^{1.2}} > 0.014 \qquad (55)$$

For a wide channel (m = 0), with M taken to be about 1.7 for a sinusoidal shaped hydrograph, I = 0.15, and E = 5, Eq. (52) becomes:

$$\frac{T_r\, S_o^{1.6}}{q_p^{0.2}\, n^{1.2}} > 0.006 \qquad (56)$$

The left-hand-side of Eq. (56) is identical with that of Eq. (55), although the right-hand-side of Eq. (55) is about two times that of Eq. (56).

Diffusion Criterion. In the case of diffusion models, Ponce et al. (1978) presented the following criterion:

$$T_D\, S_o \sqrt{g/D_o} > 30 \qquad (57)$$

Substituting Eqs. (23) and (31) into Eq. (57) allows it to be rewritten in the following equivalent form:

$$\frac{T_r\, S_o^{1.15}}{(q_p\, n)^{0.3}} > 0.0003 \qquad (58)$$

Eq. (58) can be rewritten in the following form:

$$\frac{T_r\, S_o^{0.7}\, n^{0.6}}{q_p^{0.4}} > 0.0003\ f \qquad (59)$$

where f varies from about 0.5 to 1.5 depending on a typical range of values for $S_o^{-0.5}$ and n.

Eq. (41), with E = 5, M = 1.7, I = 0.15, and m = 0, becomes:

$$\frac{T_r\, S_o^{0.7}\, n^{0.6}}{q_p^{0.4}} > 0.00024 \qquad (60)$$

Eqs. (59) and (60) are identical, although the right-hand-side of Eq. (59) can be from one-half to about two times that of Eq. (60). Thus, it is considered that this approach is in general agreement with that of Ponce et al. (1978) while being able to account for channel shape, more realistic shaped hydrographs than the sinusoidal approximation, and non-prismatic channel geometry.